CMPSCI 611: Advanced Algorithms

Lecture 24: Simplex Algorithm

Andrew McGregor

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Recap: Example Linear Program

Let

$$x_1 = \text{number of bars of Choco ordered}$$

$$x_2 = \text{number of bars of Choco Deluxe ordered}$$

Objective:

$$\max x_1 + 6x_2$$

Constraints:

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Variants of Linear Programming

Different variants:

- Objective function can be maximization or minimization.
- Constraints can be inequalities or equalities
- Variables can be restricted to be non-negative or unrestricted

Can reduce between different forms:

- Max problem to min problem: Multiply objective function by -1.
- Inequality constraints to equality constraints: Add slack variables,

$$(x_1 + x_2 + x_3 \leq 400, \quad x_2 + 3x_3 \leq 600, \quad x_1, x_2, x_3 \geq 0)$$

$$\longrightarrow (x_1+x_2+x_3+s_1 = 400, \; x_2+3x_3+s_2 = 600, \; x_1, x_2, x_3, s_1, s_2 \geq 0)$$

- Equality constraints to inequality constraints:

$$(x_1 + x_2 + x_3 = 400) \quad \longrightarrow \quad (x_1 + x_2 + x_3 \leq 400, \quad x_1 + x_2 + x_3 \geq 400)$$

Concepts

Definition

An LP is infeasible if the constraints are so tight that it is impossible to satisfy all of them. An LP is unbounded if the constraints are so loose that it is possible to achieve arbitrarily high objective values.

Definition

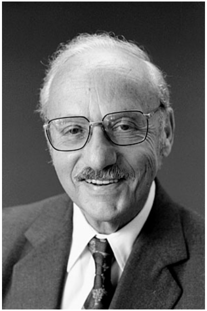
A vertex is specified by making a set of n inequalities tight. Two vertices u, v are neighbors if they have $n - 1$ defining inequalities in common.

If the linear program is feasible and bounded, the optimum is achieved at a vertex of the feasible region.

For this lecture, assume no two sets of n inequalities give the same vertex.

Simplex Algorithm

Simplex Algorithm was devised by George Dantzig in 1947...



Algorithm

Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.

Not known to be polynomial time but very quick in practice. Polynomial time algorithms do exist but are less used in practice.

Outline

Simplex in more detail

Formalizing the Simplex Algorithm

Objective:

$$\max \quad \mathbf{c}^T\mathbf{x}$$

Subject to:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

Algorithm

Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.

At each iteration there are two tasks:

- Task 1: Determine if current vertex is optimal
- Task 2: If not, determine where to move next.

Completing the tasks: Easy if current vertex is origin

Consider generic LP:

$$\max \quad \mathbf{c}^T\mathbf{x}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

If the origin $\mathbf{0}$ is feasible, it's a vertex: The n inequalities $\mathbf{x} \geq \mathbf{0}$ are tight. We'll hence forth assume $\mathbf{0}$ is feasible although there are ways around this assumption.

Lemma

The origin is optimal if and only if all $c_i \leq 0$.

Proof.

If some $c_i > 0$ then the origin is not optimal since we can increase the objective function by raising x_i. Other direction is clear. □

Hence Task 1 is easy. For Task 2, consider raising x_i as much a possible until another inequality becomes tight!

Example

Objective:

$$\max \quad 2x_1 + 5x_2$$

Constraints:

$$\begin{aligned} 2x_1 - x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 9 \\ -x_1 + x_2 &\leq 3 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

$x_1 = 0, x_2 = 0$ is a feasible solution but, e.g., we can increase x_2 until one of the other inequalities becomes tight. This happens when $x_2 = 3$.

If current vertex is not origin, shift coordinate system...

- Consider LP:

$$\max \quad \mathbf{c}^T\mathbf{x}$$

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

  and suppose current vertex $\mathbf{v} \neq \mathbf{0}$.

- Rewrite the LP using the variables

$$y_i = b_i - \mathbf{a_i} \cdot \mathbf{x}$$

$$y_j = x_j$$

  for all tight equations $x_j = 0$ or $\mathbf{a_i} \cdot \mathbf{x} = b_i$ where $\mathbf{a_i}$ are the rows of A

- In new coordinate system, current vertex is origin!

Step 1: Initial LP

Objective:

$$\max \quad 2x_1 + 5x_2$$

Constraints:

$$2x_1 - x_2 \leq 4 \quad (1)$$
$$x_1 + 2x_2 \leq 9 \quad (2)$$
$$-x_1 + x_2 \leq 3 \quad (3)$$
$$x_1 \geq 0 \quad (4)$$
$$x_2 \geq 0 \quad (5)$$

- Current Vertex: (4), (5)
- Objective Value: 0
- Increase x_2: (5) becomes loose, and (3) becomes tight at $x_2 = 3$
- New variables: $y_1 = x_1$, $y_2 = 3 + x_1 - x_2$

Step 2: Rewritten LP

Objective:

$$\max \quad 15 + 7y_1 - 5y_2$$

Constraints:

$$y_1 + y_2 \leq 7 \quad (1)$$
$$3y_1 - 2y_2 \leq 3 \quad (2)$$
$$y_2 \geq 0 \quad (3)$$
$$y_1 \geq 0 \quad (4)$$
$$-y_1 + y_2 \leq 3 \quad (5)$$

- Current Vertex: (4), (3)
- Objective Value: 15
- Increase y_1: (4) becomes loose, and (2) becomes tight at $y_1 = 1$
- New variables: $z_1 = 3 - 3y_1 + 2y_2$, $z_2 = y_2$

Step 3: Rewritten LP

Objective:

$$\max \quad 22 - 7z_1/3 - z_2/3$$

Constraints:

$$-z_1/3 + 5z_2/3 \leq 6 \quad (1)$$
$$z_1 \geq 0 \quad (2)$$
$$z_2 \geq 0 \quad (3)$$
$$z_1/3 - 2z_2/3 \leq 1 \quad (4)$$
$$z_1/3 + z_2/3 \leq 4 \quad (5)$$

- Current Vertex: (2), (3)
- Objective Value: 22
- Optimal: all $c_i < 0$ Hurray!
- Relate z_i values back to original x_i values.