CMPSCI 711 SPRING '09: HOMEWORK 1
DUE 6PM, FEBRUARY 17TH

Rules:

- Collaborations are allowed on the homework but answers must be written independently.
- Please write on your solution who you collaborated with.
- Solutions must be typeset in Latex.
- Email solutions to mcgregor@cs.umass.edu with subject "711 Homework 1 Solutions."
- If a solution is late by $h \geq 0$ hours, marks will be scaled by a factor $0.75^{h/24}$.

Questions 1 (10 marks)**.** *Write a proof or provide a counter example for each of the following statements:*

(1) *If X and Y are random variables, then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.*
(2) *If X and Y are independent random variables, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$*
(3) *If $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ for random variables X and Y then X and Y are independent.*
(4) *If X and Y are independent random variables, then $\mathbb{V}[XY] = \mathbb{V}[X]\mathbb{V}[Y]$*

Questions 2 (10 marks)**.** *n people are boarding an n seater plane and have assigned seats. Unfortunately they've* all *lost their boarding pass and everyone just takes a seat uniformly at random (from the seats available when they board).*

(1) *What's the probability that the last passenger is sitting in her correct seat?*
(2) *What's the expected number of passengers sitting in the correct seat?*
(3) *What's the variance in the number of passengers sitting in the correct seat when $n = 3$?*

Questions 3 (10 marks)**.** *Gaius Balter, a reknown scientist, has to develop an algorithm that analyzes blood samples and determines whether a sample came from a human or a cylon (a type of robot.) So far he has developed randomized algorithms with the following guarantees:*

- *Algorithm 1: The output is always "human" or "cylon." If the sample came from a human, the algorithm is always correct. If the sample came from a cylon, the algorithm is only correct with probability* 0.5*. The algorithm takes time t_1.*
- *Algorithm 2: The output is either "human", "cylon", or "fail." The algorithm fails with probability* 0.5 *but otherwise the answer returned is correct. The algorithm takes time t_2.*
- *Algorithm 3: The output is always "human" or "cylon." The algorithm returns the correct answer with probability* 0.6*. The algorithm takes time t_3.*

As Gaius' assistant, you need to use his algorithms and develop the following "improved" algorithms:

- *Based on Algorithm 1, develop an algorithm that never identifies a human as a cylon and only identifies a cylon as a human with probability* 0.001*.*
- *Based on Algorithm 2, develop an algorithm that is always correct but whose running time is random.*
- *Based on Algorithm 3, develop an algorithm that returns the correct answer with probability* 0.999*.*

Questions 4 (10 marks)**.** *Problem 4.1 from [MR] textbook.*

Questions 5 (10 marks: The Tougher Question)**.** *Given the set up of question 2, what's the probability that no passengers are sitting in their correct seats? What's the limit as n tends to ∞.*