CMPSCI 711 SPRING '09: HOMEWORK 2
DUE 6PM, MARCH 5TH

Rules:

- Collaborations are allowed on the homework but answers must be written independently.
- Please write on your solution who you collaborated with.
- Solutions must be typeset in Latex.
- Email solutions to mcgregor@cs.umass.edu with subject "711 Homework 2 Solutions."
- If a solution is late by $h \geq 0$ hours, marks will be scaled by a factor $0.75^{h/24}$.

Questions 1 (12.5 marks)**.** *In this question we prove a more general version of the Chernoff Bound than the one in the book. Let Y be a continuous random variable that takes values in the interval $[0, 1]$. Prove that for $t > 0$,*

$$\mathbb{E}\left[\exp(tY)\right] \leq 1 + \mathbb{E}\left[Y\right](e^t - 1) \ .$$

Let $X_1, \ldots, X_n$ be independent random variables that take values in the interval $[0, \alpha]$ for some $\alpha \geq 1$. Let $X = \sum_{i \in [n]} X_i$. Prove an upper bound on $\mathbb{P}\left[X > (1+\delta)\mathbb{E}\left[X\right]\right]$.

Questions 2 (12.5 marks)**.** *Let p be a prime number and let a, b be chosen uniformly at random from $\{0, 1, \ldots, p-1\}$. Define p random variables $Y_0, \ldots, Y_{p-1}$ where $Y_i = ai + b \pmod p$. Prove that these random variables are pairwise independent, i.e., for $i \neq j$ and all $\alpha, \beta \in \{0, 1, \ldots, p-1\}$:*

$$\mathbb{P}\left[Y_i = \alpha, Y_j = \beta\right] = \mathbb{P}\left[Y_i = \alpha\right]\mathbb{P}\left[Y_j = \beta\right] \ .$$

Are the random variables 3-wise independent, i.e., for distinct i, j, k and all $\alpha, \beta, \gamma \in \{0, 1, \ldots, p_1\}$, is it true that $\mathbb{P}\left[Y_i = \alpha, Y_j = \beta, Y_k = \gamma\right] = \mathbb{P}\left[Y_i = \alpha\right]\mathbb{P}\left[Y_j = \beta\right]\mathbb{P}\left[Y_k = \gamma\right]$? Let f be some function such that for all i, $f(Y_i) = 1$ with probability p and $f(Y_i) = 0$ with probability $1 - p$. Let $X = \sum_i f(Y_i)$. Prove the best upper bound you can on $\mathbb{P}\left[X = 0\right]$. (Hint: Think Chebyshev.)

Questions 3 (12.5 marks)**.** *You arrive at a mysterious island in the South Pacific along with $n-1$ other people. Let $0 \leq r \leq n$ satisfy*

$$\binom{n}{r} < 2^{\binom{r}{2} - 1} \ .$$

Using the probabilistic method prove that it is possible that in every set of r people on the island, there are at least two people who have met before and at least two people who have never met before.

Questions 4 (12.5 marks: The Tougher Question)**.** *The department is thinking about adopting a new qualifying exam for the first year students. The names of the $2n$ students (who have distinct names) are placed in $2n$ wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the students are led into the room; each may look in at most n boxes, but must leave the room exactly as she/he found it and is permitted no further communication with the others. A student passes the qualifying exam if they find their own name. Design a randomized strategy for the students such that, with probability at least 30%, all students find their own name.*