

CMPSCI 711: “Really Advanced Algorithms”

Lecture 3 – Principle of Deferred Decisions & Stable Matchings

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Outline

Clock Solitaire and Principle of Deferred Decisions

Stable Matching Problem

Probabilistic Analysis of Gale-Shapley Algorithm

Readings

Recall Last Week's Puzzle

- ▶ Take a standard pack of 52 cards that is randomly shuffled.
- ▶ Split into 13 piles of 4 and label piles $\{A, 2, \dots, 10, J, Q, K\}$.
- ▶ Take first card from "K" pile.
- ▶ Take next card from "X" pile where X is the face value of the previous card taken.
- ▶ Repeat until either all cards are removed (**you win**) or we get stuck (**you lose**).

What's the probability you win?

Structural Observations

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Proof.

- ▶ When 1st, 2nd, or 3rd K is seen we don't terminate because “ K ” pile is non-empty.
- ▶ Terminate when 4th K is seen: we win iff it's the 52nd card.



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Theorem

The probability we win clock solitaire is 1/13.

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- ▶ The matching is **unstable** if there exists w_i and m_j such that
 - ▶ w_i and m_j are not matched to each other.
 - ▶ w_i prefers m_j to her match.
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Does a stable matching always exist? Can we find one in polynomial time?

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Does the algorithm terminate? Is the resulting matching stable?

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- ▶ Once a woman becomes matched she doesn't become unmatched (although she may change her match.)
- ▶ All the woman to which m_i proposed are already matched.
- ▶ If m_i has proposed to everyone, all the women are matched, hence all the men are matched. Contradiction!



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- ▶ At each stage of the algorithm, let t_i be the number of women to which m_i could still potentially propose.
- ▶ At each step $\sum_{i \in [n]} t_i$ decreases by 1.
- ▶ Initially $\sum_{i \in [n]} t_i = n^2$ so there can be at most n^2 steps.



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- ▶ Since m_i prefers w_l to w_j , he must have proposed to w_l before he proposed to w_j .
- ▶ But then, w_l must prefer her current match to m_i : either she already had a better match when m_i proposed or she matched m_i initially and then got a better proposal. Contradiction!



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- ▶ In **probabilistic analysis**, we consider random input and investigate what happens when it’s processed by a fixed algorithm. E.g., the Gale-Shapley algorithm when the preference lists are random.

Theorem

If the preference lists are random, the expected number of iterations of Gale-Shapley is $\leq nH_n$.

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 - ▶ The expected running time of the modified algorithm is an upper bound for the running time of original algorithm.

Probabilistic Analysis of Gale-Shapley Algorithm (2/2)

Theorem

If the preference lists are random, the expected number of iterations of Gale-Shapley is at most nH_n .

Proof.

Since the algorithm terminates once all women have received at least one proposal, the random process is analogous to the coupon collector problem. □

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For next time, please make sure you've read:

- ▶ Appendix C: Basic Probability Theory (9 pages)
- ▶ Chapter 1: Introduction up to section 1.4 (14 pages)