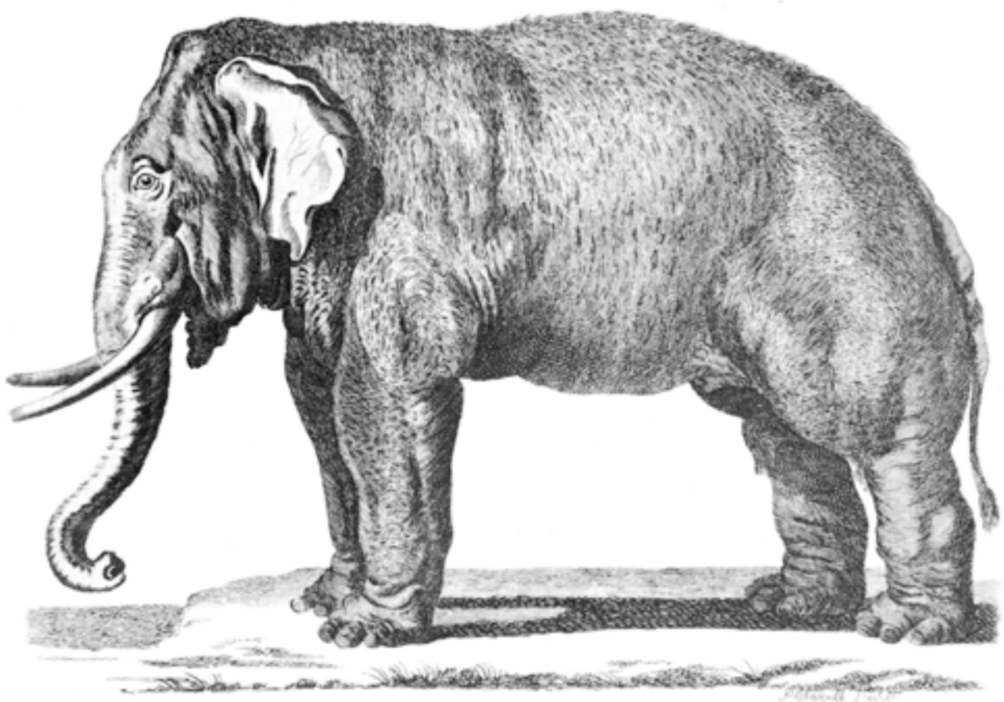
Checking & Spot-Checking the Correctness of Priority Queues

Matthew Chu & Sampath Kannan (UPenn) Andrew McGregor (UCSD)



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- *Related Work:*

Program Checking	[Blum, Kannan '95]
Memory Checking	[Blum et al. '94]
Checking linked Data Structures	[Amato, Loui '94]

Priority Queues

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- *Example:* Insert 5, Insert 4, Extract-min, Insert 7,... would correspond to the sequence (5,1), (4,2), (4,2), (7,4), ... if the *PQ* was correct.

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- *Constraints:* The interaction sequence is observed as a stream and has limited space.
- We are interested in *offline* checkers that identify errors by the end of the interaction sequence.

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- <u>*Checkers:*</u>

  A randomized, offline, $O(\sqrt{n} \log n)$-space checker that identifies errors with prob. $1-1/n$.

  Any randomized, offline checker of a "certain type" requires $\Omega(\sqrt{n})$ space.

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- ***Spot-Checker:***

  A randomized, offline, $O(\varepsilon^{-1} \log^2 n)$-space spot-checker that identifies a priority queue that is "ε-far" from correct with prob. $1-1/n$.

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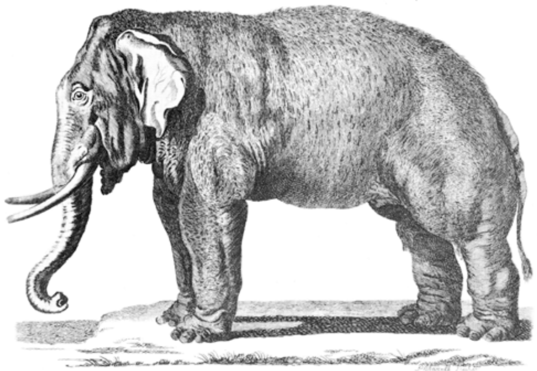
2: Checking
3: Spot-Checking



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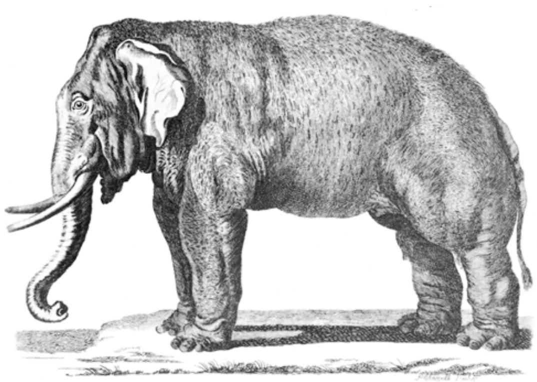

Correctness

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- *Thm:* An interaction sequence is correct iff it satisfies:

  $C1$: $\{(u,t)\}=\{(u,t)\}$

  $C2$: For all $c_s=(u,t)$: t<s

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  $((u,ta) < (v,sa))$ then $(sb<ta$ or $tb<sa)$

- *Proof Idea:* If correct then clearly $C1$, $C2$, & $C3$. For other direction consider first incorrect extract-min...

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  $$\Pr[h(x) = h(y)] \leq \delta \qquad \text{if } x \neq y.$$

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- *What it means for us:*
  Let x_t be (u,t) if u was inserted at time t
  Let y_t be (u,t) if an extract returns (u,t)
  Hence can easily check CI: {(u,t)}={(u,t)}

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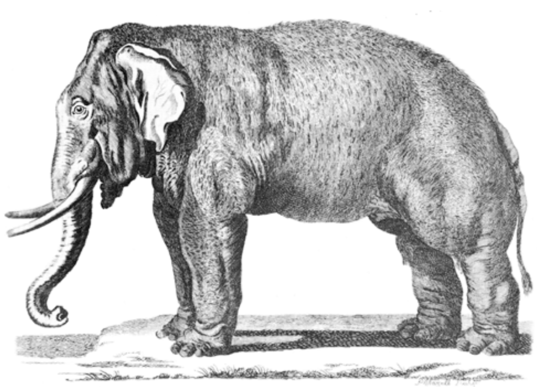

Checking Results

- *Thm:* A randomized, offline, $O(\sqrt{n} \lg n)$-space checker that identifies errors with prob. 1-1/n.
- *Thm:* Any randomized online checker that is correct with prob. 3/4 requires $\Omega(n/\lg n)$ space.
- *Thm:* Any deterministic offline checker requires $\Omega(n)$ space.
- Outline why $\Omega(\sqrt{n})$ space looks necessary for randomized, offline checkers...

Algorithm Intuition

- *Key Idea:* $c_{ta}=(u,t)$ should imply that all elements inserted before *ta* and not extracted are greater than c_{ta}

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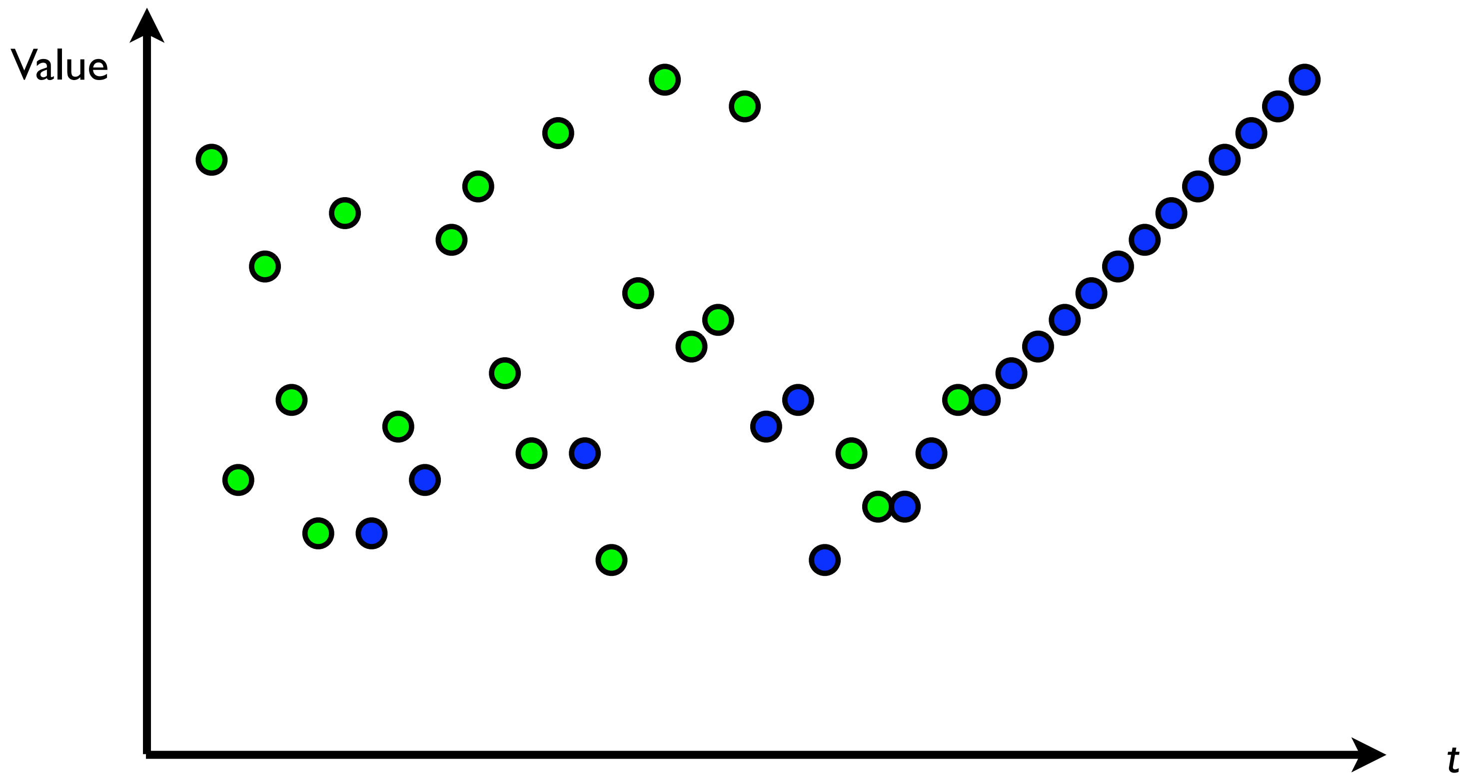
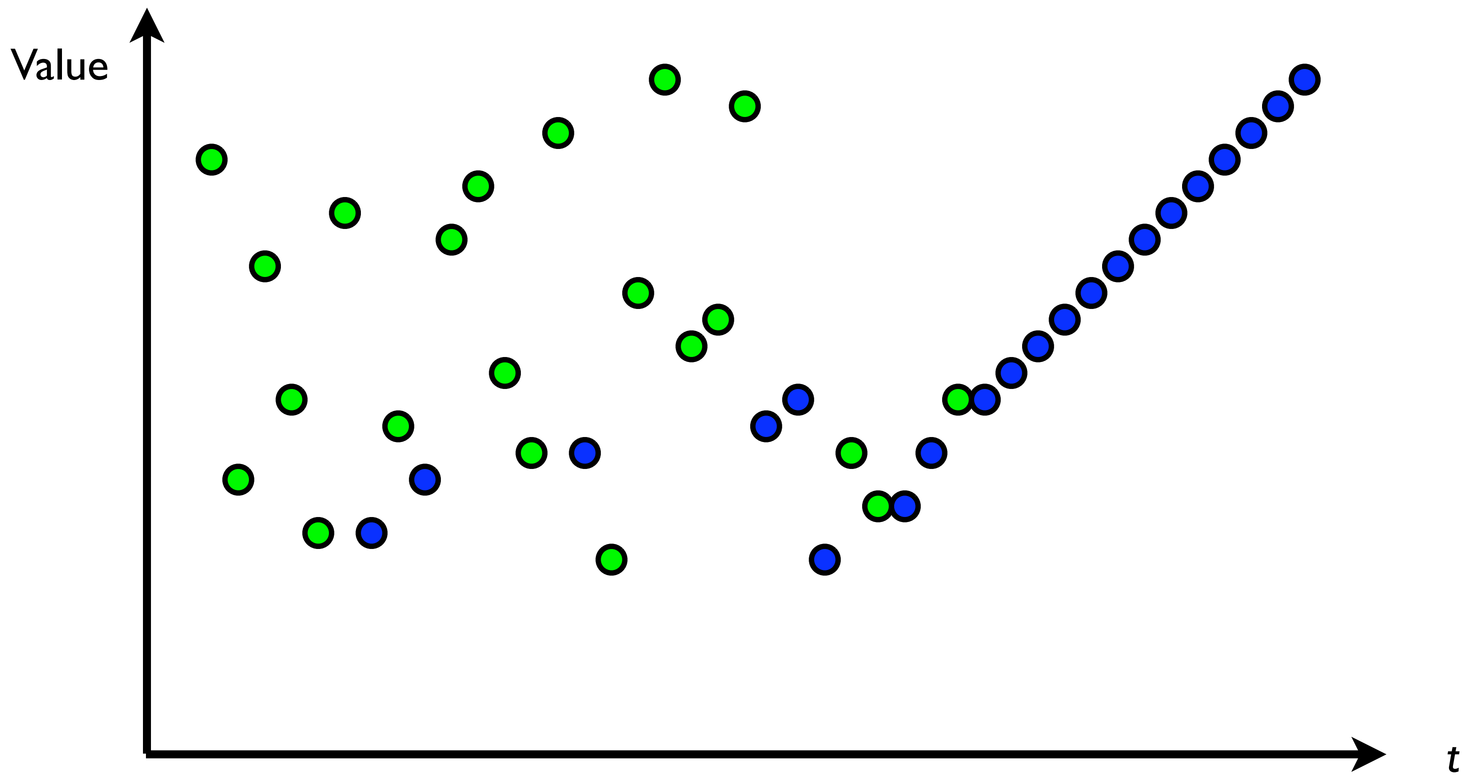


Algorithm Outline

- *Split* sequence into $\sqrt{n}$-length *Epochs*
- *Identify* errors within present epoch immediately
- *Maintain* lower-bound on contents of past epochs.



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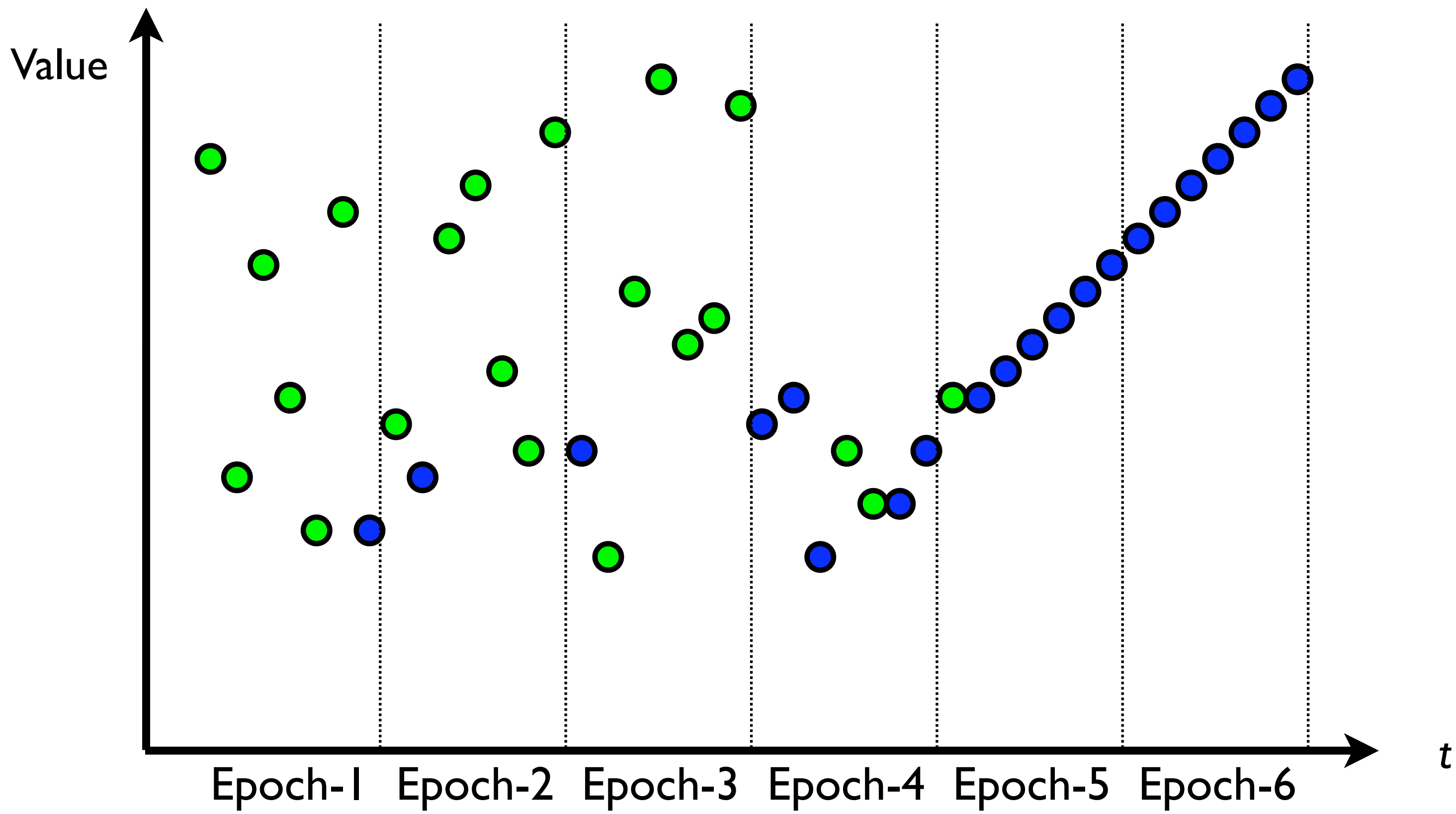

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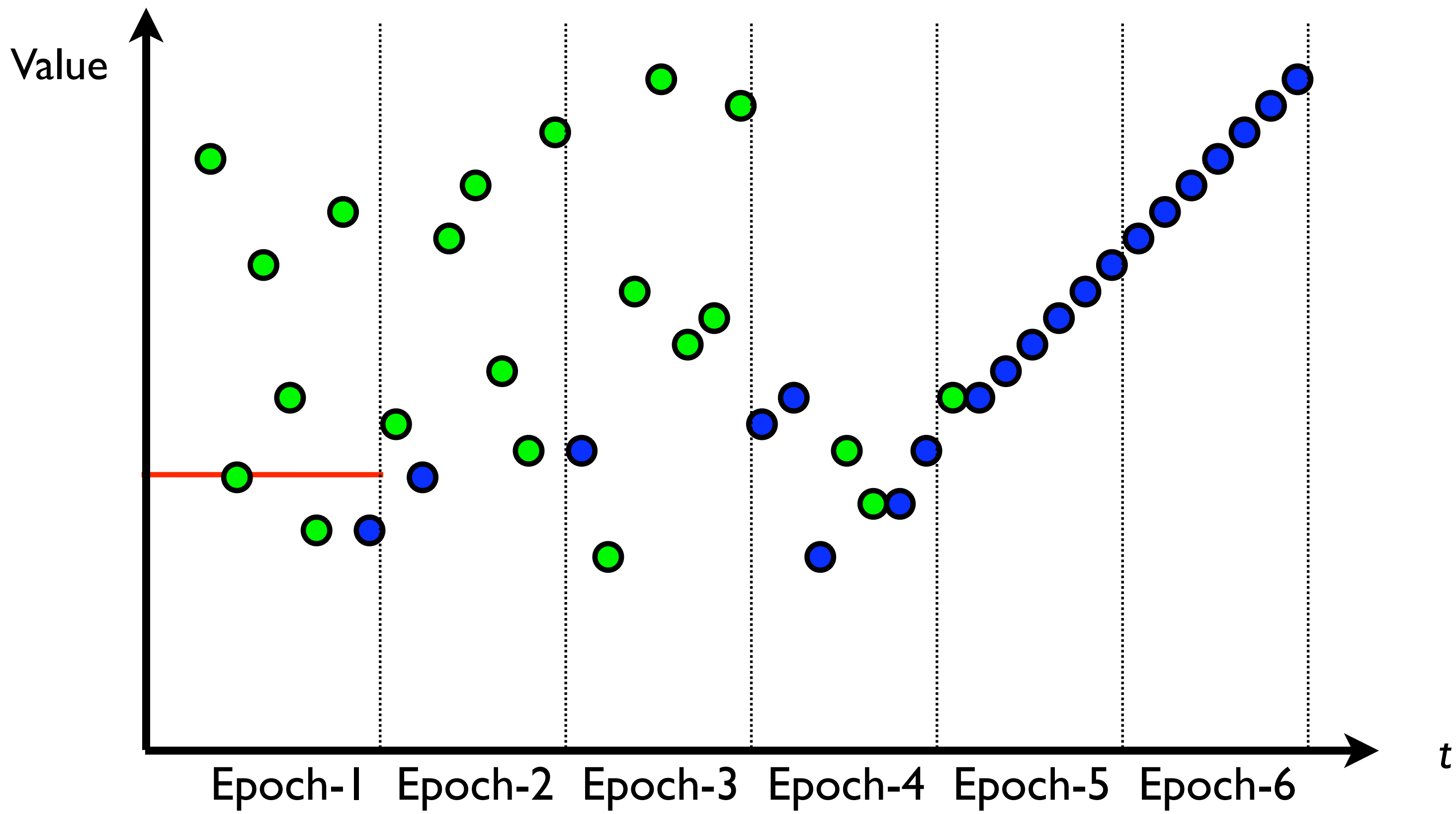

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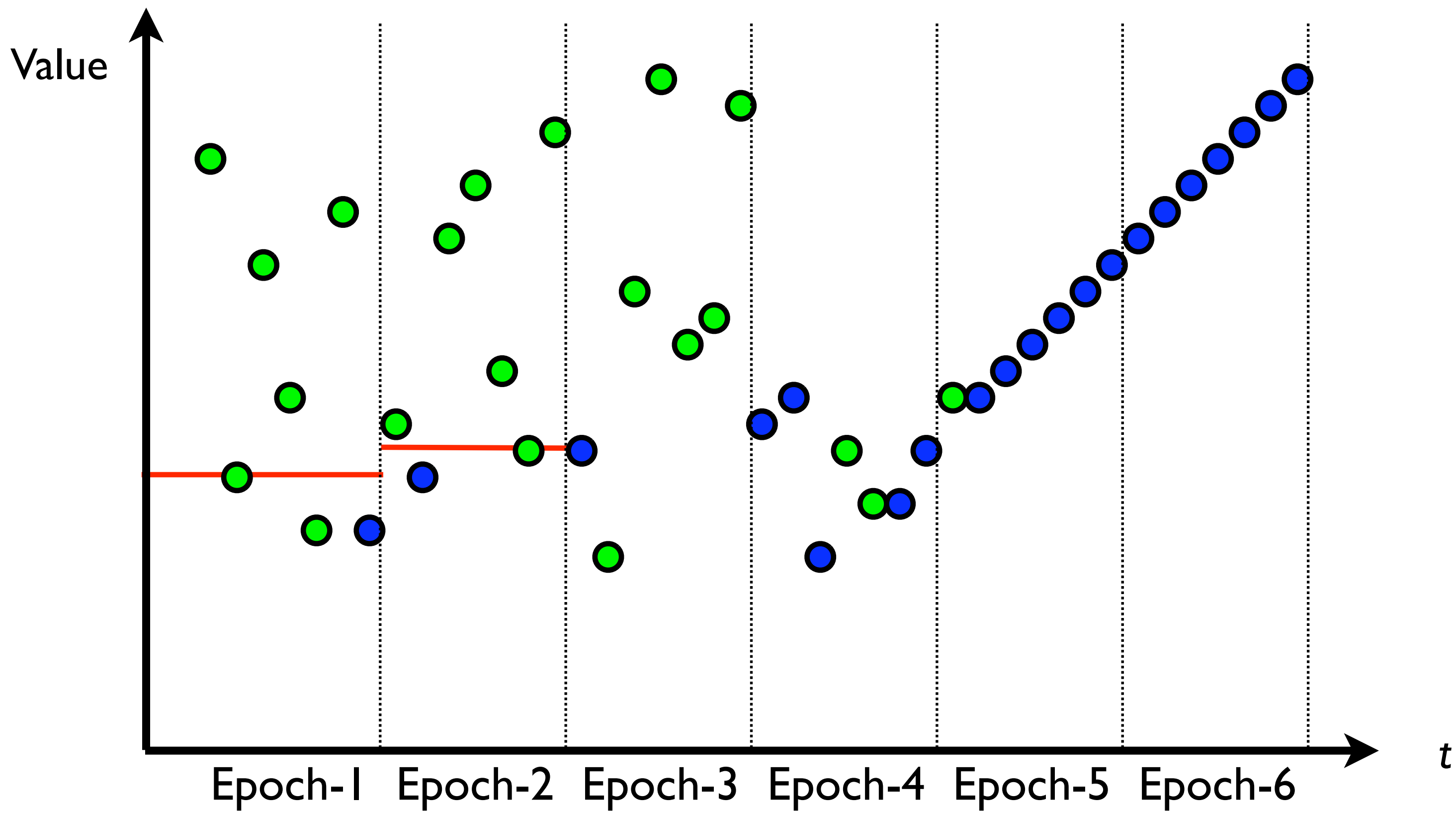

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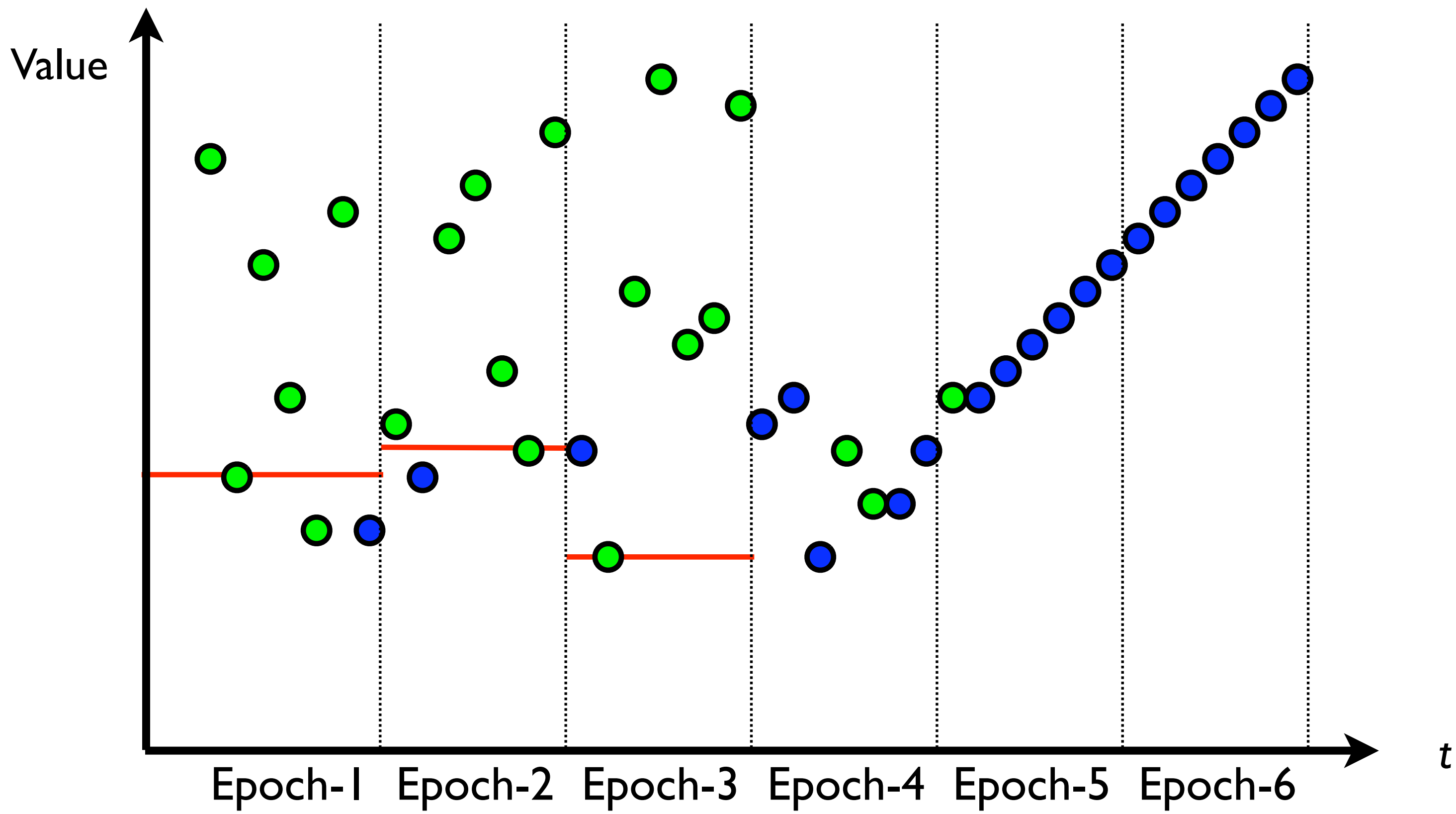

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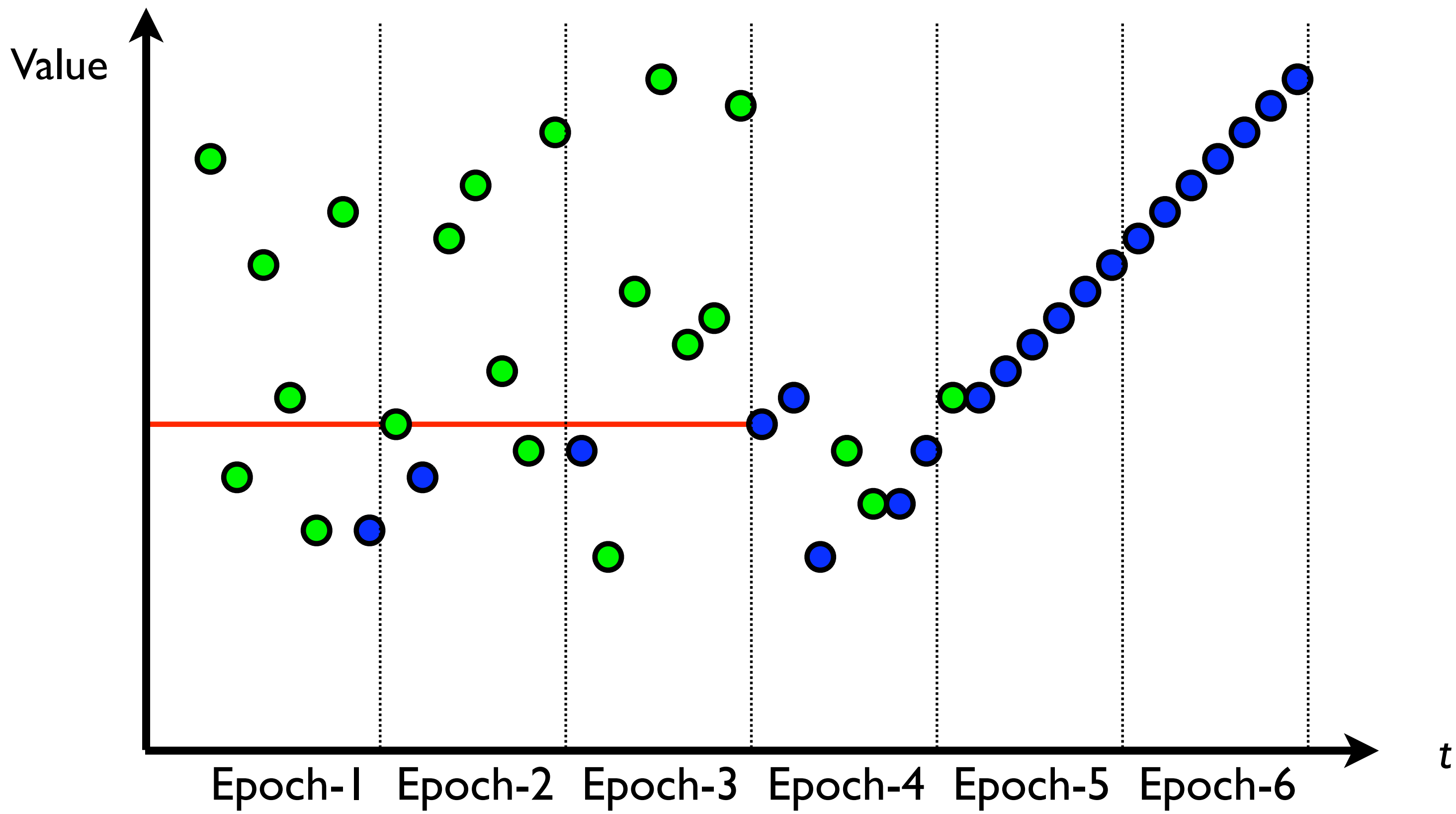

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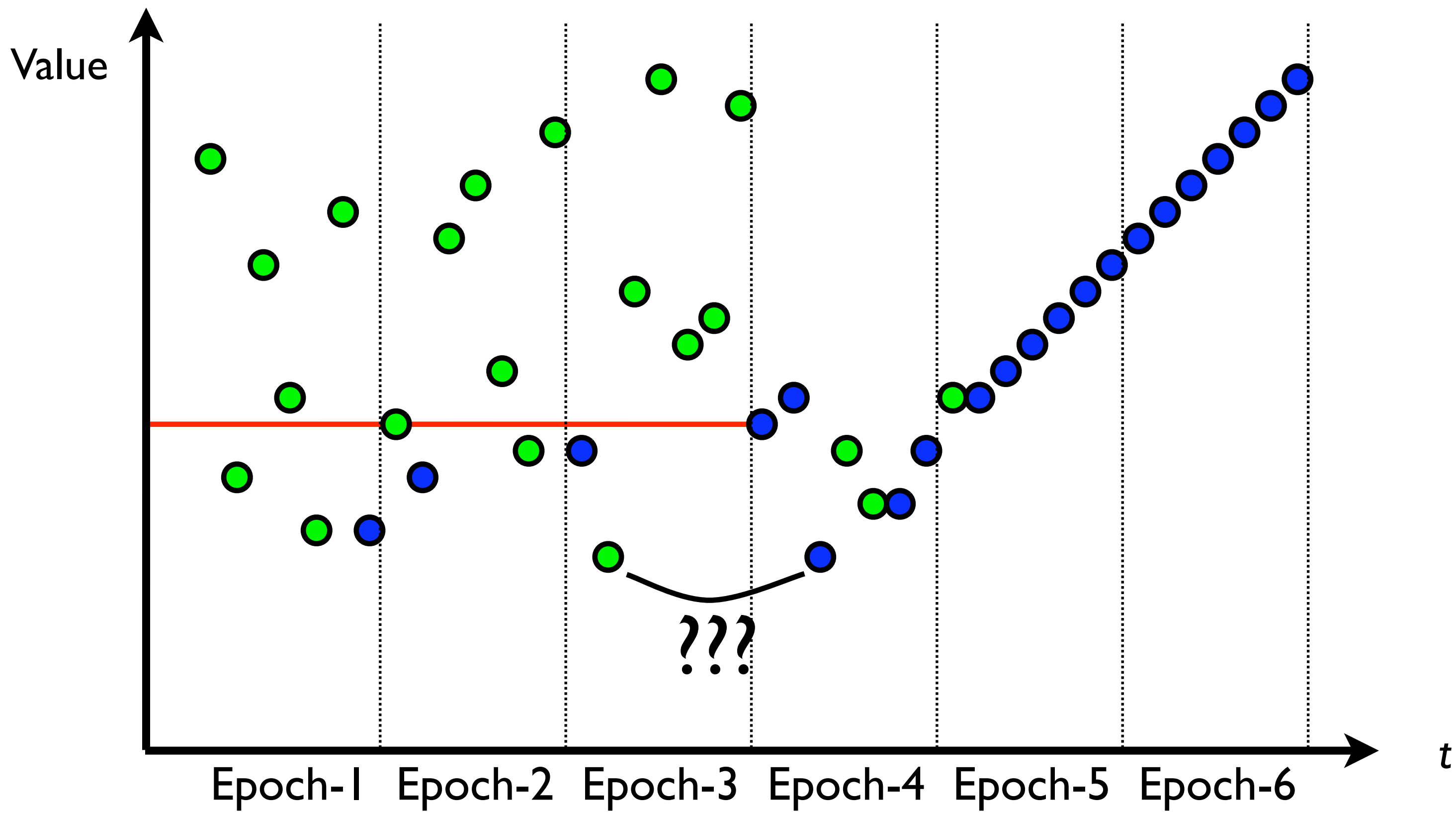

Algorithm Detail

```
For k in [2√n], let f(k)=0
For i=1 to 2√n:
  Let Buffer be empty
  For j in Epoch-i={(i-1)√n+1,...,i√n}:
    If c_i=(u,t), add c_i to B
    If c_i=(u,t):
      If t in Epoch-k (k<i) and f(k)>c_i then FAIL!
      If t in Epoch-i and c_i > min Buffer then FAIL!
      Remove c_i from Buffer (if present)
      For k<i, let f(k)=max(f(k),c_i)
    Let f(i)=min Buffer
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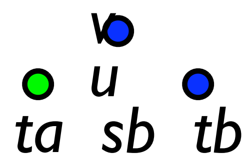

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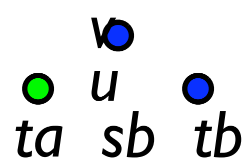

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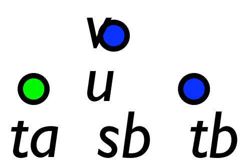

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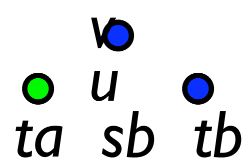

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- *Case 2:* If $i<j$ then $f(i)\geq(v,sb)$ and hence we fail at time *tb* (or before.)

Online or Deterministic?

- *Thm:* Any online checker that is correct with prob. 3/4 requires $\Omega(n/\lg n)$ space.
- *Thm:* Any offline deterministic checker requires $\Omega(n)$ space.

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binary string x

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& index i in $[n]$

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"Is the length i prefix of x and y equal?"

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[Chakrabarti, Cormode, McGregor '07]

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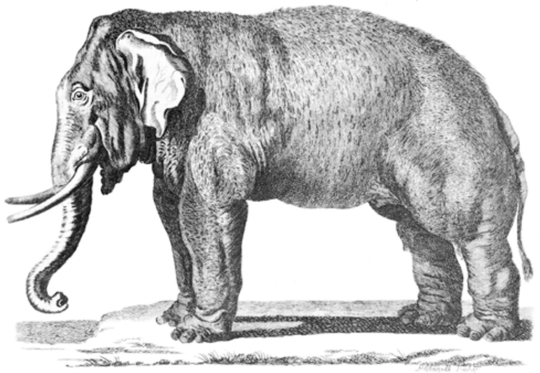

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- Consider interaction sequence $c_1, \ldots, c_{2n}$ and perm. π of [2n]. Define new interaction sequence $d_1, \ldots, d_{2n}$ where

  $d_{\pi(i)} = (u, \pi(i))$ if $c_i = (\mathrm{u}, \mathrm{i})$

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- Say interaction sequence $c_1, \ldots, c_{2n}$ is ε-far if no permutation with less than εn rearrangements results in a correct interaction sequence.

Revealing Tuples

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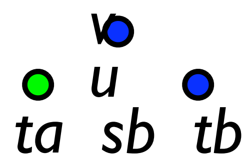

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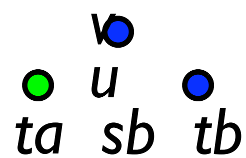

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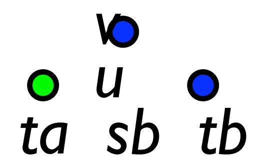

- *Thm:* An interaction sequence that is ε-far from being correct has at least εn revealing tuples.

- *Proof:*

  Find first incorrect extract-min, say $c_{sb}=(v,sa)$.

  Since this isn't minimum element, there exists (u,ta) and $c_{tb}=(u,ta)$ such that $ta<sb<tb$.

  Moving tb to sb reduces # of revealing tuples.

  Continue until sequence is correct.

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  Will identify $c_{sb}=(v,sa)>(u,ta)$ and $c_{tb}=(u,ta)$ such that $ta<sb<tb$.

Summary

- ***Checkers:***

  A randomized, offline, $O(\sqrt{n} \log n)$-space checker that identifies errors with prob. $1-1/n$.

  Any randomized, offline checker of a "certain type" requires $\Omega(\sqrt{n})$ space.

  Online or deterministic requires $\Omega(n)$ space.

- ***Spot-Checker:***

  A randomized, offline, $O(\varepsilon^{-1} \lg^2 n)$-space spot-checker that identifies a priority queue that is "ε-far" from correct with prob. $1-1/n$.

- *... and that's how you mind you P.Q.'s!*