

1. (Based on [MR95], problem 4.1):

Suppose you are given a biased coin that has $\Pr[HEADS] = h$, for $a \leq h \leq b$, for some fixed a and b . You are not given any other information about h (i.e., you can not assume that it is chosen randomly).

- (a) Using Chebyshev's inequality, devise a procedure for estimating h by a value \hat{h} such that you can guarantee that $\Pr[|h - \hat{h}| > \rho h] < \mu$, for any choice of the constants $0 < a, b, \rho, \mu < 1$. Let N be the number of times you need to flip the biased coin to obtain the estimate, where N is a function of a, b, ρ , and μ . What is the smallest value of N for which you can still give this guarantee?
- (b) In Lecture 16, we saw the following two Chernoff bounds:

$$\Pr[B(n, p) \leq (1 - \delta)np] \leq e^{-\delta^2 np/2}, \text{ and}$$

$$\Pr[B(n, p) \geq (1 + \delta)np] \leq e^{-\delta^2 np/3},$$

where $B(n, p)$ is a random variable representing the number of heads seen in n tosses of a coin that is heads with probability p . Using these bounds, what is the smallest value of N for which you can still give the guarantee that $\Pr[|h - \hat{h}| > \rho h] < \mu$?

2. CLR, Problem 36-1 (Page 961).

3. The decision version of the clique problem can be stated as follows:

INPUT: An undirected graph G and an integer k .

QUESTION: Does G contain a clique of size k ?

Prove that if we can solve the decision version of the clique problem in polynomial time, then we can find the largest clique in any graph in polynomial time.

4. A Boolean formula is in k -CNF form if it is in conjunctive normal form, and has exactly k literals in each clause. We can thus define, for each k , the problem k -SAT:

INPUT: A Boolean formula Φ in k -CNF.

QUESTION: Is Φ satisfiable?

- (a) Using the fact that 3-SAT is NP-Complete, show that k -SAT is NP-Complete, for any $k > 3$.

- (b) Prove that 2-SAT is not NP-Complete unless $P=NP$.

(Hint: construct a directed graph, where vertices correspond to literals in Φ , and edges correspond to "implications" from the clauses. The graph should have the property that Φ is satisfiable if and only if the graph does not contain a directed path of a certain form.)