Assignment 0 (Sol.)

Reinforcement Learning

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Note: This is an ungraded assignment. Marks scored in this assignment will **not** be counted towards the final score.

1. There are n bins of which the kth contains $k-1$ blue balls and $n-k$ red balls. You pick a bin at random and remove two balls at random without replacement. Find the probability that:

   - the second ball is red;
   - the second ball is red, given that the first is red.

   (a) 1/3, 2/3

   (b) 1/2, 1/3

   (c) 1/2, 2/3

   (d) 1/3, 1/3

   Sol. (c)
   Let C_i be the colour of the ith ball. In each bin, there are a total of $(k-1)+(n-k) = (n-1)$ balls. Of these half are blue and the other half are red (verify $\sum_{k=1}^{n} k-1 = \sum_{k=1}^{n} n-k$).

   The probability of the second ball being red is equal to the probability of the second ball being red given that the first ball was either red or blue.

   For a particular bin we have,

   $$P(C_2 = red) = \frac{(n-k)}{(n-1)}\frac{(n-k-1)}{(n-2)} + \frac{(k-1)}{(n-1)}\frac{(n-k)}{(n-2)} = \frac{n-k}{n-1}$$

   Considering all bins, we have

   $$P(C_2 = red) = \sum_{k=1}^{n} \frac{n-k}{n(n-1)} = \frac{1}{2}$$

   The probability of the second ball being red given that the first ball was red,

   $$P(C_2 = red|C_1 = red) = \frac{P(C_2 = red, C_1 = red)}{P(C_1 = red)}$$

   Now, $P(C_1 = red) = \frac{1}{2}$.

For a particular bin,

$$P(C_2 = red, C_1 = red) = \frac{(n-k)}{(n-1)} \frac{(n-k-1)}{(n-2)}$$

Considering all bins, we have

$$P(C_2 = red | C_1 = red) = \sum_{k=1}^{n} \frac{\frac{(n-k)(n-k-1)}{n(n-1)(n-2)}}{\frac{1}{2}}$$

Simplifying, we have

$$P(C_2 = red | C_1 = red) = \frac{2}{3}$$

2. A medical company touts its new test for a certain genetic disorder. The false negative rate is small: if you have the disorder, the probability that the test returns a positive result is 0.999. The false positive rate is also small: if you do not have the disorder, the probability that the test returns a positive result is only 0.005. Assume that 2% of the population has the disorder. If a person chosen uniformly from the population is tested and the result comes back positive, what is the probability that the person has the disorder?

   (a) 0.803
   (b) 0.976
   (c) 0.02
   (d) 0.204

   Sol. (a)
   Let,

   - T be the probability of the test being positive,
   - D be the probability of a person having the disorder

   From the data provided, we have:

   - $P(T|D) = 0.999$
   - $P(T|\neg D) = 0.005$
   - $P(D) = 0.02$
   - $P(\neg D) = 1 - P(D) = 0.98$

   We want to calculate the probability of a person chosen uniformly at random having the disorder given that the test came back positive, i.e.,

$$P(D|T)$$

   Now from the Bayes' theorem, we have

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\neg D)P(\neg D)}$$

   Substituting known values, we have

$$P(D|T) = \frac{0.999 * 0.02}{0.999 * 0.02 + 0.005 * 0.98} = 0.803$$

3. In an experiment, n coins are tossed, with each one showing up heads with probability p independently of the others. Each of the coins which shows up heads is then tossed again. What is the probability of observing 5 heads in the second round of tosses, if we toss 15 coins in the first round and $p = 0.4$?

   (Hint: First find the mass function of the number of heads observed in the second round.)

   (a) 0.372
   (b) 0.055
   (c) 0.0345
   (d) 0.0488

   Sol. (b)
   The same result will be observed if we toss each of the coins twice and count the number of coins showing two consecutive heads. The probability of showing two consecutive heads, given that the probability of showing heads in a toss is p, is equal to p^2. Hence, the number of heads X observed after the second round of tosses can be given by

$$P(X = r) = \binom{n}{x} p^{2r}(1 - p^2)^{n-r}$$

   Now substituting the given values into the above pmf we have,

$$P(X = 5) = \binom{15}{5} 0.4^{10}(1 - 0.4^2)^{10} = 0.055$$

4. An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

   (a) 0.5101
   (b) 0.81
   (c) 0.6308
   (d) 0.7405

   Sol. (d)
   The probability that there will be a seat available for every passenger who shows up is equal to the probability that less than or equal to 50 passengers show up. This is the same as 1 minus the probability that exactly 52 or 51 passengers show up. Thus, the required probability

$$= 1 - 0.95^{52} - 52(0.95)^{51}(0.05) = 0.7405.$$

5. Let X have mass function

$$f(x) = \begin{cases} \{x(x+1)\}^{-1} & \text{if x = 1, 2, ...,} \\ 0 & \text{otherwise,} \end{cases}$$

   and let $\alpha \in \mathbb{R}$. For what values of α is it the case that $\mathbb{E}(X^\alpha) < \infty$?

(a) $\alpha < \frac{1}{2}$

(b) $\alpha < 1$

(c) $\alpha > 1$

(d) $\alpha > \frac{3}{4}$

Sol. (b)
We have

$$E[X^\alpha] = \sum_{x=1}^{\infty} \frac{x^\alpha}{x(x+1)}$$

This expression is finite only if $\alpha < 1$.

6. Is the following a distribution function?

$$F(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

If so, give the corresponding density function. If not, mention why it is not a distribution function.

(a) No, not a monotonic function

(b) Yes, $x^{-2}e^{-1/x}, x > 0$

(c) No, not right continuous

(d) Yes, $x^{-1}e^{-1/x}, x > 0$

Sol. (b)
$F(x) \to 1$ as $x \to \infty$ and $F(x) = 0$ for $x < 0$ by definition. The monotonicity and continuity of F follow from the corresponding properties of $e^{-1/x}$. Also, since all continuous functions are right continuous, F is right continuous. Thus, the given function is a distribution function. Its corresponding density function is given by differentiating:

$$f(x) = \begin{cases} x^{-2}e^{-1/x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

7. Let $A^{m \times n}$ be a matrix of real numbers. The matrix AA^T has an eigenvector x with eigenvalue b. Then the eigenvector y of A^TA which has eigenvalue b is equal to

(a) x^TA

(b) A^Tx

(c) x

(d) Cannot be described in terms of x

Sol. (b)

$$(AA^T)x = bx$$

Multiplying by A^T on both sides and rearranging,

$$(A^T)(AA^Tx) = A^T(bx)$$

$$(A^TA)(A^Tx) = b(A^Tx)$$

Hence, A^Tx is an eigenvector of A^TA, with eigenvalue b.

8. Let $A^{n\times n}$ be a row stochastic matrix - in other words, all elements are non-negative and the sum of elements in every row is 1. Let b be an eigenvalue of A. Which of the following is true?

   (a) $|b| > 1$

   (b) $|b| <= 1$

   (c) $|b| >= 1$

   (d) $|b| < 1$

   Sol. (b)
   Note that $Ax = bx$ where x is an eigenvector and b is an eigenvalue. Let x_{max} be the largest element of x. Let A_{ij} denote the i^{th} row, j^{th} column element of A, and x_j denote the j^{th} element of x. Now,

$$\sum_{j=1}^{j=n} A_{ij}x_j = bx_i$$

   Let us consider the case where x_i corresponds to the maximum element of x. The RHS is equal to bx_{max}. Now, since $\sum_{j=1}^{j=n} A_{ij} = 1$ and $A_{ij} > 0$, the LHS is less than or equal to x_{max}. Hence,

$$bx_{max} \leq x_{max}$$
$$|b| \leq 1$$

9. Let u be a $n \times 1$ vector, such that $u^Tu = 1$. Let I be the $n \times n$ identity matrix. The $n \times n$ matrix A is given by $(I - kuu^T)$, where k is a real constant. u itself is an eigenvector of A, with eigenvalue -1. What is the value of k?

   (a) -2

   (b) -1

   (c) 2

   (d) 0

   Sol. (c)

$$(I - kuu^T)u = -u$$
$$u - ku(u^Tu) = -u$$
$$2u - ku = 0$$

   Hence, k = 2

10. Which of the following are true for any $m \times n$ matrix A of real numbers

   (a) The rowspace of A is the same as the columnspace of A^T

   (b) The rowspace of A is the same as the rowspace of A^T

   (c) The eigenvectors of AA^T are the same as the eigenvectors of A^TA

   (d) The eigenvalues of AA^T are the same as the eigenvalues of A^TA

Sol. (a) & (d)
Since the rows of A are the same as the columns of A^T, the rowspace of A is the same as the columnspace of A^T . The eigenvalues of AA^T are the same as the eigenvalues of A^TA, because if $AA^Tx = \lambda x$ we get $A^TA(A^Tx) = \lambda(A^Tx)$. (b) is clearly not necessary. (c) need not hold, since although the eigenvalues are same, the eigenvectors have a factor of A^T multiplying them.

11. The Singular Value Decomposition (SVD) of a matrix R is given by USV^T. Consider an orthogonal matrix Q and $A = QR$. The SVD of A is given by $U_1S_1V_1^T$. Which of the following is/are true?

    (a) $U = U_1$
    (b) $S = S_1$
    (c) $V = V_1$

    Sol. (b) & (c)
    The matrix V_1 represents the eigenvectors of A^TA. Now

    $$A^TA = (R^TQ^T)(QR)$$

    Since Q is orthogonal, $Q^TQ = I$. Therefore,

    $$A^TA = (R^TIR)$$

    $$A^TA = (R^TR)$$

    Since these matrices are equal, their eigenvectors will be the same. V represents the eigenvectors of R^TR and V_1 the eigenvectors of A^TA. Hence, $V = V_1$. Also, since S represents the eigenvalues of A^TA (as well as AA^T , since the set of eigenvalues is the same for both), $S = S_1$. However, U need not be equal to U_1 , since $AA^T = (QR)(R^TQ^T) = Q(RR^T)Q^T$.