CMPSCI 240: "Reasoning Under Uncertainty"

Lecture 7

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Reminders

- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- Second homework is due on Friday

Recap

Last Time: Common Discrete Random Variables

- Uniform: $p_X(k) = 1\,/\,(b - a + 1)$ for $k = a, a+1, \ldots, b$

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- Poisson: $p_X(k) = e^{-\lambda}\, \lambda^k\, /\, k!$ for $k = 0, 1, 2, \ldots$

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- A weighted average of the possible values of X
- The value we'd "expect" to get for X "on average" if we repeated the same experiment (and calculated X) many times
- Useful if we want a single number that "summarizes" p_X

Last Time: Expectations of Standard Random Variables

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- Poisson random variable X: $\mathbb{E}[X] = \lambda$

Expectation (Cont.)

Expected Value Rule and Linearity of Expectation

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- e.g., if $Y = aX + b$ for any scalars a and b, what is $\mathbb{E}[Y]$?

Examples of Linearity of Expectation

- e.g., Amherst's temperature is modeled by a random variable X with mean $\mathbb{E}[X]$ equal to 10 degrees Celsius. What is the mean temperature expressed in degrees Farenheit?

Variance

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- Variance of X measures the dispersion of X around $\mathbb{E}[X]$:

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- Standard deviation of X is

$$\sigma_X = \sqrt{\text{var}(X)}$$

Examples of Variance

- e.g., on any given day, with probability 0.2, I am not especially tired and will drink only 3 cups of coffee before work. With probability 0.8, however, I am VERY tired and will drink 6 cups of coffee before work. Let X be the number of cups that I will drink. $\mathbb{E}[X] = 5.4$. What is $\text{var}(X)$?

Variance of Standard Random Variables

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- e.g., var(X) and σ_X for a Bernoulli random variable?
- e.g., var(X) and σ_X for a Poisson random variable?

Variance of a Linear Function of a Random Variable

- Let $Y = aX + b$ for any random variable X and scalars a and b. $\mathbb{E}[Y] = a\,\mathbb{E}[X] + b$. What is $\mathrm{var}(Y)$? What about σ_Y?

Examples of Linearity of Variance

- e.g., Amherst's temperature is modeled by a random variable X with mean $\mathbb{E}[X]$ and standard deviation σ_X both equal to 10 degrees Celsius. A day is "typical" if it is within one standard deviation of the mean. What is the temperature range for a "typical" day expressed in degrees Farenheit?

Multiple Random Variables

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- e.g., choosing a random faculty member in the department: X is the person's height, Y is the person's weight

Tabular Representation

	y_1	y_2	y_3	y_4
x_1	0.1	0.1	0	0.2
x_2	0.05	0.05	0.1	0
x_3	0	0.1	0.2	0.1

- e.g., $p_{X,Y}(x_2, y_3) = 0.1$, $p_{X,Y}(x_3, y_1) = 0$, ...

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- e.g., $p_{X,Y}(x_2, y_3) = 0.1$, $p_{X,Y}(x_3, y_1) = 0$, ...
- Is this a valid joint PMF for X and Y? How do we know?

Marginal PMFs

	y_1	y_2	y_3	y_4
x_1	0.1	0.1	0	0.2
x_2	0.05	0.05	0.1	0
x_3	0	0.1	0.2	0.1

- We can compute the PMFs of X and Y from $p_{X,Y}$

$$p_X(x) = \sum_y p_{X,Y}(x, y) \text{ and } p_Y(y) = \sum_x p_{X,Y}(x, y)$$

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$$\mathbb{E}[g(X, Y)] = \sum_{x} \sum_{y} g(x, y)\, p_{X,Y}(x, y)$$

- e.g., what is $\mathbb{E}[aX + bY + c]$ for any scalars a, b, and c?

Three or More Random Variables

- The joint PMF of X, Y, and Z is denoted $p_{X,Y,Z}$ where

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- $\mathbb{E}[aX + bY + cZ + d] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] + c\,\mathbb{E}[Z] + d$

[Got to here in class...]

Examples of Three or More Random Variables

- e.g., an office party decides to do a gift exchange for their annual holiday party. Everyone in the office purchases a gift and places it in a box. Each person is then given a random gift drawn from the box. What is the expected number of people who get back the gift that they purchased?

Expectations of Standard Random Variables

- e.g., if X is a binomial random variable, what is $\mathbb{E}[X]$?

For Next Time

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