CMPSCI 240: “Reasoning Under Uncertainty”

Lecture 8

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Reminders

- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- Second homework is due TOMORROW

Recap

Last Time: Expectation

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- If $Y = a\,X + b$ then $\mathbb{E}[Y] = a\,\mathbb{E}[X] + b$

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- If $Z = aX + bY + c$ then $\mathbb{E}[Z] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] + c$

Multiple Random Variables (Cont.)

Three or More Random Variables

- The joint PMF of X, Y, and Z is denoted $p_{X,Y,Z}$ where

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- The expected value of $g(X, Y, Z)$ is

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- $\mathbb{E}[aX + bY + cZ + d] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] + c\,\mathbb{E}[Z] + d$

Linearity of Expectation

- In general, the expectation of a sum of random variables is equal to the sum of their expectations, i.e.,

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$$\mathbb{E}[X_1 + \ldots + X_n] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

- If X_i has the same PMF as X_j they are identically distributed:

$$\mathbb{E}[X_i] = \sum_{x} x\, p_{X_i}(x) = \sum_{x} x\, p_{X_j}(x) = \mathbb{E}[X_j]$$

Examples of Three or More Random Variables

- e.g., an office party decides to do a gift exchange for their annual holiday party. Everyone in the office purchases a gift and places it in a box. Each person is then given a random gift drawn from the box. What is the expected number of people who get back the gift that they purchased?

Expectations of Standard Random Variables

- e.g., if X is a binomial random variable, what is $\mathbb{E}[X]$?

Conditioning

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- Conditional PMF of X given Y: denoted by $p_{X|Y}$, where

$$p_{X|Y}(x \mid y) = P(X = x \mid Y = y) = P(\{X = x\} \mid \{Y = y\})$$

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- Compute $p_{X|Y}$ using the definition of conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \implies p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

Conditioning and the Tabular Representation

	y_1	y_2	y_3	y_4
x_1	0.1	0.1	0	0.2
x_2	0.05	0.05	0.1	0
x_3	0	0.1	0.2	0.1

- Compute conditional PMF for Y given $X = x$ by renormalizing the values in the row for x and conditional PMF for X given $Y = y$ by renormalizing the values in the column for y

Conditioning and the Tabular Representation

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- e.g., to compute $p_{X|Y}(x_1 \mid y_2)$:

$$p_{X|Y}(x_1 \mid y_2) = \frac{p_{X,Y}(x_1, y_2)}{\sum_{i=1}^{3} p_{X,Y}(x_i, y_2)} = \frac{p_{X,Y}(x_1, y_2)}{p_Y(y_2)}$$

Calculating Joint and Marginal PMFs

- Apply the multiplication rule using the underlying events $\{X = x\} \cap \{Y = y\}$, $\{X = x\}$, and $\{Y = y\}$ to give

$$p_{X,Y}(x, y) = p_X(x)\, p_{Y|X}(y \mid x) = p_Y(y)\, p_{X|Y}(x \mid y)$$

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- Divide-and-conquer approach to calculating marginal PMFs:

$$p_X(x) = \sum_y p_{X,Y}(x, y) = \sum_y p_Y(y)\, p_{X|Y}(x \mid y)$$

Examples of Conditioning

- e.g., if my computer has a virus (which is true with probability 0.2) my antivirus program will detect this virus with probability 0.7. If my computer does not have a virus, my antivirus program will mistakenly detect a virus with probability 0.1. Let X and Y be binary random variables that are equal to 1 if my computer has a virus and the program detects a virus, respectively, and 0 otherwise. What is $p_{X,Y}$?

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- e.g., $X =$ "sum of two rolls", $A =$ "first roll is odd"
- Nothing fancy here – this is just notation – when in doubt, think about the underlying events and outcomes

Examples of Conditioning on Events

- e.g., a student will take a test repeatedly up to a maximum of n times, each time with a probability p of passing, independent of previous attempts. What is the PMF of the number of attempts, given that the student passes the test?

Conditional Expectation

- Conditional version of expectation:

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- Expectations can be conditioned on events too

Conditional Variance and Standard Deviation

- Conditional variance of X given $Y = y$:

$$\text{var}(X \mid Y = y) = \mathbb{E}[X^2 \mid Y = y] - \mathbb{E}[X \mid Y = y]^2$$

Conditional Variance and Standard Deviation

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$$\sigma_{X \mid Y = y} = \sqrt{\text{var}(X \mid Y = y)}$$

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- What is the conditional standard deviation of X given $Y = y$?

$$\sigma_{X \mid Y = y} = \sqrt{\text{var}(X \mid Y = y)}$$

- Both can be conditioned on events too

Total Expectation Theorem

- Total expectation theorem: for any random variables X and Y

$$\mathbb{E}[X] = \sum_y p_Y(y)\,\mathbb{E}[X \mid Y = y]$$

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- For any disjoint events $A_1, \ldots, A_n$ that partition Ω:

$$\mathbb{E}[X] = \sum_{i=1}^{n} P(A_i)\,\mathbb{E}[X \mid A_i]$$

Expectations of Standard Random Variables

- e.g., if X is a geometric random variable, what is $\mathbb{E}[X]$?

Independence

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- If $P(A) > 0$ this is equivalent to $P(B \,|\, A) = P(B)$

Independence of Random Variables

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- Equivalently if for all x and y such that $p_Y(y) > 0$

$$\frac{p_{X,Y}(x,y)}{p_Y(y)} = p_{X|Y}(x \,|\, y) = p_X(x)$$

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- Equivalently if for all x and y such that $p_Y(y) > 0$

$$\frac{p_{X,Y}(x,y)}{p_Y(y)} = p_{X|Y}(x \mid y) = p_X(x)$$

- Or if $p_{Y|X}(y \mid x) = p_Y(y)$ for all x and y such that $p_X(x) > 0$

Independence and the Tabular Representation

	y_1	y_2	y_3	y_4
x_1	0.05	0.15	0	0.2
x_2	0.025	0.075	0	0.1
x_3	05	0.15	0	0.2

- Are X and Y independent? How can we tell?

Expectation and Independence

- If X and Y are independent then

$$\begin{aligned}\mathbb{E}[XY] &= \sum_x \sum_y x\, y\, p_{X,Y}(x, y) \\ &= \sum_x \sum_y x\, y\, p_X(x)\, p_Y(y) = \mathbb{E}[X]\, \mathbb{E}[Y]\end{aligned}$$

Expectation and Independence

- If X and Y are independent then

$$\mathbb{E}[XY] = \sum_x \sum_y x\, y\, p_{X,Y}(x,y)$$
$$= \sum_x \sum_y x\, y\, p_X(x)\, p_Y(y) = \mathbb{E}[X]\,\mathbb{E}[Y]$$

- This is only true if X and Y are independent

Independence and Three or More Random Variables

- X, Y, and Z are independent if and only if for all x, y, and z

$$p_{X,Y,Z}(x,y,z) = p_X(x)\, p_Y(y)\, p_Z(z)$$

Independence and Three or More Random Variables

- X, Y, and Z are independent if and only if for all x, y, and z

$$p_{X,Y,Z}(x, y, z) = p_X(x)\, p_Y(y)\, p_Z(z)$$

- If random variables X, Y, and Z are independent then so are random variables of the form $f(X)$, $g(Y)$, and $h(Z)$

Variance and Independence

- If $X_1, X_2, \ldots, X_n$ are independent then

$$\text{var}(X_1 + X_2 + \ldots + X_n) = \sum_{i=1}^{n} \text{var}(X_i)$$

Variance and Independence

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$$\text{var}(X_1 + X_2 + \ldots + X_n) = \sum_{i=1}^{n} \text{var}(X_i)$$

- If $X_1, X_2, \ldots, X_n$ are independent and identically distributed, then $\text{var}(X_1) = \text{var}(X_2) = \ldots = \text{var}(X_n)$ and

$$\text{var}(X_1 + X_2 + \ldots + X_n) = n\,\text{var}(X_1)$$

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