CMPSCI 240: "Reasoning Under Uncertainty"

Lecture 13

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Reminders

- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- Fifth homework is due on Friday

The Markov and Chebyshev Inequalities

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- Use the mean (and variance) of a random variable X to draw conclusions about the probabilities of various events
- Useful where the mean (and variance) of X can be computed easily, but the PMF is unavailable or hard to compute

The Markov Inequality

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- Intuition: if a nonnegative random variable X has a small mean, then the probability that X is large is small

Examples of the Markov Inequality

- e.g., suppose a class of students gets an average score of 60% on a test. Find an upper bound on the probability that a randomly-chosen student's score is 72% or higher.

Examples of the Markov Inequality

- e.g., suppose you fill in your tax return with random integers from \$0 to \$9,999 (do not do this!!!) and your accountant says that your expected rebate is \$50. Find an upper bound on the probability that you get at least \$500 back. Find an lower bound on the probability that you get less than \$100 back.

Examples of the Markov Inequality

- e.g., suppose you randomly draw a card from a deck of cards (with replacement) until you draw the ace of spades. Use the Markov inequality to find an upper bound on the probability that it takes ≥ 100 draws to draw the ace of spades. What is the actual probability that it takes ≥ 100 draws?

Examples of the Markov Inequality

- e.g., suppose X can take on the values $0, 1, \ldots, 4$ with equal probabilities. What is the PMF of X? What is the mean of X? Use the Markov inequality to find upper bounds on $P(X >= 2)$, $P(X >= 3)$, and $P(X >= 4)$. What are the corresponding exact probabilities? What can you conclude about the bounds provided by the Markov inequality?

Limitations of the Markov Inequality

- Bounds provided by the Markov inequality can be quite loose
- Can obtain tighter bounds if we use the variance as well

The Chebyshev Inequality

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- Note that X does not need to be nonnegative

Examples of the Chebyshev Inequality

- e.g., suppose the average salary of a CS professor is \$83k per year and the standard deviation is \$20k. Bound the probability that a CS professor makes $\geq$ \$123k or $\leq$ \$43k.

Examples of the Chebyshev Inequality

- e.g., suppose the average speed of any driver is 49 mph and the variance is 144mph. Bound the probability that the speed of a random driver is between 29 and 69 mph exclusive.

Examples of the Chebyshev Inequality

- e.g., suppose X can take on the values $0, 1, \ldots, 4$ with equal probabilities and $\mathbb{E}[X] = 2$. Use the Chebyshev inequality to bound the probability that $|X - 2| \geq 1$.

[Got to here in class...]

The Weak Law of Large Numbers

The Weak Law of Large Numbers

- If $X_1, X_2, \ldots, X_n$ are independent identically-distributed random variables with mean $\mathbb{E}[X_i]$, then for every $\epsilon > 0$

$$P\left(\left|\frac{X_1 + \ldots + X_n}{n} - \mathbb{E}[X_i]\right| \geq \epsilon\right) \to 0 \quad \text{as } n \to \infty$$

- Intuition: the sample mean of a very large number of independent identically distributed random variables is very close to the true mean with high probability

Examples of the Weak Law of Large Numbers

- e.g., suppose we would like to estimate the president's approval rating p. We ask n random voters whether or not they approve of the president and use the fraction of voters who approve as our estimate. If we would like to have high confidence (e.g., 95%) that our estimate is very accurate (i.e., within 0.01 of the true approval rating) how many voters should we poll? Hint: suppose $p(1-p) \leq 1/4$

For Next Time

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